

p. 110 [1.1]

Prove that $a^2 + ab + b^2 \geq 0$ and state when equalities hold

Proof.

$$\begin{aligned} a^2 + ab + b^2 &= \left(a + \frac{b}{2}\right)^2 + b^2 - \left(\frac{b}{2}\right)^2 \\ &= \left(a + \frac{b}{2}\right)^2 + \frac{3b^2}{4} \\ &\geq 0 \end{aligned} \tag{1}$$

$\therefore a^2 + ab + b^2 \geq 0$. The equality holds when $a = b = 0$.

□

Discussion. Once you obtain (1), you are essentially done. The square of any number is non-negative and the sum of non-negative numbers is a non-negative number. When you must show an expression is non-negative, re-writing it as a sum of squares (if possible) is a good strategy.

p. 110 [1.2]

Prove that $5x^2 - 4xy + 6y^2 \geq 0$ and state when equalities hold

Proof.

$$\begin{aligned} 5x^2 - 4xy + 6y^2 &= 5\left(x^2 - \frac{4xy}{5}\right) + 6y^2 \\ &= 5\left(x - \frac{2y}{5}\right)^2 - \frac{4y^2}{5} + 6y^2 \\ &= 5\left(x - \frac{2y}{5}\right)^2 + \frac{26y^2}{5} \\ &\geq 0 \end{aligned}$$

$\therefore 5x^2 - 4xy + 6y^2 \geq 0$. The equality holds when $x = y = 0$.

□

Discussion. As in the previous proof, rewriting the expression as the sum of squares enabled us to conclude the expression is non-negative.

p. 110 [1.3]

Prove that $x^2 + y^2 \geq 4x - 6y - 13$ and state when equalities hold

Proof.

$$\begin{aligned}x^2 + y^2 - 4x + 6y + 13 &= x^2 - 4x + y^2 + 6y + 13 \\&= (x - 2)^2 + (y + 3)^2 + 13 - 4 - 9 \\&= (x - 2)^2 + (y + 3)^2 \\&\geq 0\end{aligned}$$

$\therefore x^2 + y^2 \geq 4x - 6y - 13$. The equality holds when $x = 2$, $y = -3$.

□

Discussion. As in the proof of [1.1], rewriting the expression as the sum of squares enabled us to conclude the expression is non-negative.

p. 111 [2]

Prove that $(a^2 + b^2)(x^2 + y^2) \geq (ax + by)^2$.

Proof.

$$\begin{aligned}(a^2 + b^2)(x^2 + y^2) - (ax + by)^2 &= a^2 x^2 + b^2 x^2 + a^2 y^2 + b^2 y^2 - a^2 x^2 - 2abxy + b^2 y^2 \\&= b^2 x^2 + 2abxy + a^2 y^2 + 2b^2 y^2 \\&= (bx + ay)^2 + 2b^2 y^2 \\&\geq 0\end{aligned}$$

$\therefore (a^2 + b^2)(x^2 + y^2) \geq (ax + by)^2$.

□

Discussion. As in the proofs on page 110, rewriting the expression as the sum of squares enabled us to conclude the expression is non-negative.

p. 112 [3]

Prove that $\sqrt{a} - \sqrt{b} < \sqrt{a-b}$, for $a > b > 0$.

Proof.

Suppose $a > b > 0$.

$$\begin{aligned}
 (\sqrt{a} - \sqrt{b})^2 - (\sqrt{a-b})^2 &= a - 2\sqrt{a}\sqrt{b} + b - a + b \\
 &= 2b - 2\sqrt{a}\sqrt{b} && (1) \\
 &< 2b - 2\sqrt{b}\sqrt{b} && (2) \\
 &= 2b - 2b \\
 &= 0
 \end{aligned}$$

Since $(\sqrt{a} - \sqrt{b})^2 - (\sqrt{a-b})^2 < 0$, $(\sqrt{a} - \sqrt{b})^2 < (\sqrt{a-b})^2$.

$\therefore \sqrt{a} - \sqrt{b} < \sqrt{a-b}$.

QED

Discussion. The overall strategy of this proof is to demonstrate that $(\sqrt{a} - \sqrt{b})^2 < (\sqrt{a-b})^2$, then use THM 1 to conclude that $\sqrt{a} - \sqrt{b} < \sqrt{a-b}$. We show $(\sqrt{a} - \sqrt{b})^2 < (\sqrt{a-b})^2$ by showing that the difference $(\sqrt{a} - \sqrt{b})^2 - (\sqrt{a-b})^2$ is less than zero. The key step is at (2). By replacing \sqrt{a} in (1) by the smaller number \sqrt{b} , the term $2\sqrt{b}\sqrt{b}$ in (2) is smaller than the term $2\sqrt{a}\sqrt{b}$ in (1). Since at (2), a smaller quantity is subtracted from $2b$ than is subtracted from $2b$ in (1), the expression $2b - 2\sqrt{b}\sqrt{b}$ in (2) is greater than the expression $2b - 2\sqrt{a}\sqrt{b}$ in (1). Note that the conditions for using THM 1 are satisfied. {You should review THM 1 and verify this.}

p. 113 [4.1]

Prove that $a + \frac{1}{a} \geq 2$, for $a > 0$, $b > 0$. State when the equality holds.

Proof.

Suppose $a > 0$, $b > 0$. The inequality $\frac{a+b}{2} \geq \sqrt{ab}$, which says that the arithmetic mean is no less than the geometric mean, is true for all non-negative numbers a and b . So is it certainly true when b equals $\frac{1}{a}$, which is non-negative because we have assumed $a > 0$. Thus,

$$a + b \geq 2\sqrt{ab} \implies a + \frac{1}{a} \geq 2\sqrt{a \frac{1}{a}} \implies a + \frac{1}{a} \geq 2.$$

$\therefore a + \frac{1}{a} \geq 2$ for $a > 0$, $b > 0$. The equality holds when $a = b = 1$.

QED

Discussion. Why does the theorem require that $a > 0$ rather than allowing $a \geq 0$? Because, the division by a at $a + \frac{1}{a} \geq 2$ would be meaningless were $a = 0$. This example shows the utility of remembering that a statement that is true for all numbers is true for each particular number, including a particular number that is convenient for whatever purpose you have in mind.

p. 113 [4.2]

Prove that $(\frac{a}{b} + \frac{c}{d})(\frac{b}{a} + \frac{d}{c}) \geq 4$, for $a > 0$, $b > 0$, $c > 0$, $d > 0$. State when the equality holds.

Proof.

Suppose $a > 0$, $b > 0$, $c > 0$, $d > 0$. The inequality $\frac{A+B}{2} \geq \sqrt{AB}$, which says that the arithmetic mean is no less than the geometric mean, is true for all non-negative numbers A and B . So is it certainly true when A equals $\frac{a}{b}$, and B equals $\frac{c}{d}$ which are all non-negative because we have assumed $a > 0$, $b > 0$, $c > 0$, $d > 0$. The relation $(A + B)(\frac{1}{A} + \frac{1}{B}) \geq 4$ has been shown to follow from the theorem of the arithmetic and geometric means. Thus $(\frac{a}{b} + \frac{c}{d})(\frac{b}{a} + \frac{d}{c}) \geq 4$.

$\therefore (\frac{a}{b} + \frac{c}{d})(\frac{b}{a} + \frac{d}{c}) \geq 4$ for $a > 0$, $b > 0$, $c > 0$, $d > 0$. The equality holds when $a = b = c = d = 1$.

QED

Discussion. As soon as one notices that $\frac{b}{a}$ and $\frac{d}{c}$ are inverses of $\frac{a}{b}$ and $\frac{c}{d}$ respectively, the proof is obvious. Especially so, if one recalls the demonstration at pps 112-113 of J10.

p. 113 [5]

Prove that $\sqrt{x} + \sqrt{y} \leq \sqrt{2(x+y)}$, for $x > 0, y > 0$.

Proof.

Suppose $x > 0, y > 0$.

$$(\sqrt{x} + \sqrt{y})^2 - (\sqrt{2(x+y)})^2 = x + 2\sqrt{x}\sqrt{y} + y - 2x - 2y$$

$$= 2\sqrt{xy} - x - y \tag{1}$$

$$\leq x + y - x - y \tag{2}$$

$$= 0$$

Since $(\sqrt{x} + \sqrt{y})^2 - (\sqrt{2(x+y)})^2 \leq 0$, $(\sqrt{x} + \sqrt{y})^2 \leq (\sqrt{2(x+y)})^2$. The expressions that are squared are each positive, so $(\sqrt{x} + \sqrt{y})^2 \leq (\sqrt{2(x+y)})^2 \implies \sqrt{x} + \sqrt{y} \leq \sqrt{2(x+y)}$.

$\therefore \sqrt{x} + \sqrt{y} \leq \sqrt{2(x+y)}$ for $x > 0, y > 0$.

QED

Discussion. The key step uses theorem of the arithmetic and geometric means to rewrite (1) as (2).

p. 114 [6]

Prove the **triangle inequality** $|x + y| \leq |x| + |y|$ by comparing the $|x + y|^2$ and $(|x| + |y|)^2$.

Proof.

$$|x + y|^2 - (|x| + |y|)^2 = x^2 + 2xy + y^2 - |x|^2 - 2|x||y| - |y|^2$$

$$= 2xy - 2|x||y| \tag{1}$$

$$\leq 2xy - 2xy \tag{2}$$

$$= 0$$

Since $|x + y|^2 \leq (|x| + |y|)^2$ and both squared expressions are positive, $|x + y| \leq |x| + |y|$

$$\therefore |x + y| \leq |x| + |y|$$

QED

Discussion. The proof is straightforward once one recalls that $|A|^2 = A^2$ and that $A \leq |A|$.

$2xy - 2|x||y|$ is at least as great as $2xy - 2|x||y|$, because $2xy$ is no greater than $2|x||y|$. The triangle inequality is important. It is used frequently in calculus and in mathematics generally. It may be understood as a statement about distance: the distance between two points is not more than the sum of the distances from the two to any third point.

p. 114 [7]

Prove the inequality $|x| - |y| \leq |x + y|$

Proof.

$$\begin{aligned} (|x| - |y|)^2 - |x + y|^2 &= |x|^2 - 2|x||y| + |y|^2 - x^2 - 2xy - y^2 \\ &= -2xy - 2|x||y| && (1) \\ &\leq 2xy - 2xy && (2) \\ &= 0 \end{aligned}$$

Since $(|x| - |y|)^2 \leq |x + y|^2$ and both squared expressions are positive, $|x| - |y| \leq |x + y|$

$$\therefore |x| - |y| \leq |x + y|$$

QED

Discussion. Same comments as previous proof. Do note the relationship established by the last two proofs $|x| - |y| \leq |x + y| \leq |x| + |y|$. What does it tell us about distance?